On Kreimer's Hopf algebra structure of Feynman graphs

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Abstract. We reinvestigate Kreimer's Hopf algebra structure of perturbative quantum field theories with a special emphasis on overlapping divergences. Kreimer first disentangles overlapping divergences into a linear combination of disjoint and nested ones and then tackles that linear combination by the Hopf algebra operations. We present a formulation where the Hopf algebra operations are directly defined on any type of divergence. We explain the precise relation to Kreimer's Hopf algebra and obtain thereby a characterization of their primitive elements.

1 Introduction

This paper is the result of our efforts to understand the article by Dirk Kreimer on the Hopf algebra structure of perturbative quantum field theories [1]. That article was brought to our attention by Alain Connes in his talk during the Vietri conference on noncommutative geometry. Kreimer discovered that divergent Feynman graphs can be understood as elements of a Hopf algebra. The forest formula guiding the renormalization of Feynman graphs with subdivergences is reproduced by a certain interplay of product, coproduct, antipode and counit of that Hopf algebra. Meanwhile Connes and Kreimer elaborated a deep structural link [2] between that Hopf algebra of renormalization and the Hopf algebra emerging in the computation of the local index formula for transverse hypoelliptic operators [3]. This indicates that renormalization provides a mathematical calculus that can be thought of as a refinement of diffeomorphisms.

As explained by Kreimer in [1,2,4] and in private discussions, overlapping divergences require a special treatment. Overlapping divergences must first be disentangled into a linear combination of terms containing disjoint or nested divergences exclusively. Suppose the (divergent) integrand $I(q_1, \ldots, q_n)$ corresponding to a Feynman graph depends on n external parameters (masses and momenta) $q_i, \ldots, q_n$. The idea is to write

$$I(q_1, q_2, \ldots, q_n) = \{I(q_1, q_2, \ldots, q_n) - I(q_1, 0, \ldots, 0)\} + I(q_1, 0, \ldots, 0) \,.$$

The integrand $\{I(q_1, q_2, \ldots, q_n) - I(q_1, 0, \ldots, 0)\}$ is less divergent, in the optimal case convergent or without subdivergences. It is therefore sufficient to consider integrands depending on a single scale q. In the same way as above one can write $I(q) = \{I(q) - I'(q)\} + I(q)$, where $I'(q)$ is derived from $I(q)$ by nullifying q in some parts of $I(q)$. It was shown in [5] that by this procedure (which is encoded in the Schwinger-Dyson equation) it is always possible to disentangle overlapping divergences. Hence one can restrict the operations of the Hopf algebra to terms containing no overlapping divergences. The forest formula is trivial in this case, it simply says that the subdivergences must be compensated in ascending order.

In this paper we present our independent approach to the problem of overlapping divergences. Our goal is to treat overlapping divergences on the same footing with disjoint and nested ones so that the operations of the Hopf algebra are directly defined on any Feynman graph. We show that this aim can be achieved by endowing Kreimer's parenthesized words (PW) describing the Feynman graphs with additional information. In our formulation, a PW is a collection of all maximal forests of a Feynman graph, where identical regions in various forests are visualized. We show that one of the antipode axioms recovers the forest formula in its full beauty for any Feynman graph. Following an idea by Dirk Kreimer [2,4] we describe the precise relation between his and our formulations of the Hopf algebra of renormalization. In this way we gain an explicit construction of those primitive elements of Kreimer's Hopf algebra which are different from the graphically primitive elements.

Our paper is organized as follows: We introduce in Sect. 2 our extended PWs and discuss in Sect. 4 the R-operation of renormalization. The Hopf algebra is identified in Sect. 5, where longer proofs are delegated to the appendix. In Sect. 6 we discuss the relation to the Hopf algebra of Kreimer. In Sects. 3 and 7 we apply our methods

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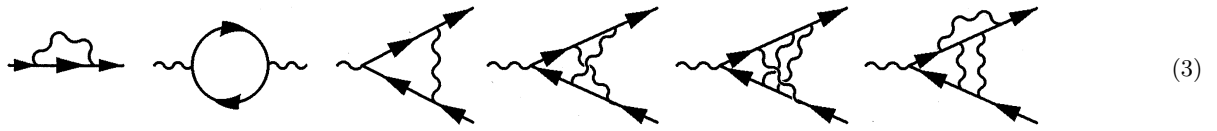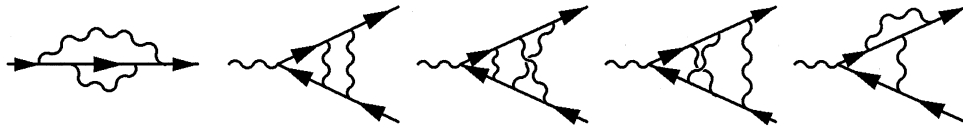[b] and Universität Leipzig, raimar@cpt.univ-mrs.fr, supported by the German Academic Exchange Service (DAAD), grant no. D/97/20386
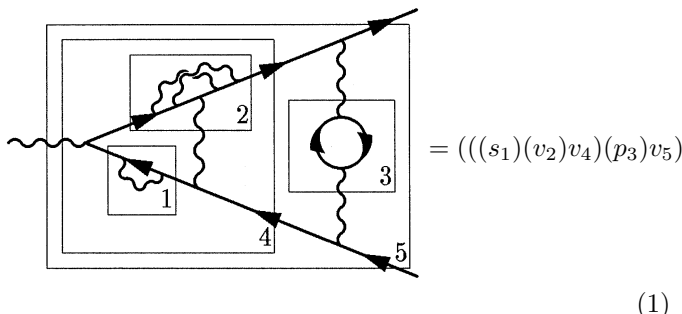*Correspondence to*: R. Wulkenhaar

$$(3)$$

$$(4)$$

to examples of Feynman graphs with overlapping divergences.

2 Feynman graphs, maximal forests and parenthesized words

Let Γ be a Feynman graph. In the way described by Kreimer we draw boxes around superficially (UV-) divergent sectors of Γ:

$$= (((s_1)(v_2)v_4)(p_3)v_5)$$

$$(1)$$

(As usual, straight lines stand for fermions and wavy lines for bosons.) A superficially divergent sector [6] is necessarily a region of Γ which contains loops. The boxes must be drawn in such a way that no vertex of Γ is situated on the border of the box and no line of Γ is tangential to the border. Boxes can be deformed. During the deformation, no vertex is allowed to pass the border and at no time a line may be tangent to the border of the box. We consider boxes which differ by a deformation as identical.

We shall work in four dimensional spacetime, but generalization is obvious. A criterion for superficial divergence of a region confined in a box is power counting. The box under consideration will contain n_B bosonic and n_F fermionic external legs. Ghosts are regarded as bosons here. In a renormalizable theory there can only be a superficial (ultraviolet) divergence in the box if it contains at least one loop and if the power counting degree of divergence d_{pc} satisfies

$$d_{pc} := 4 - n_B - (3/2)n_F \geq 0 . \qquad (2)$$

Owing to symmetries the actual degree of divergence d of one graph or a sum of graphs can be lower than d_{pc} calculated from (2), see [6]. Examples are graphs in QED with

$n_B = 3$, $n_F = 0$ (which can be omitted due to Furry's theorem) and with $n_B = 4$, $n_F = 0$ (which are superficially convergent due to gauge symmetry). Always if $d < 0$ the box must be erased. This does not mean that there cannot be divergences in the box to erase. But these non-superficial divergences must be contained in other boxes which cannot be deformed into the box we erased.

Our boxes represent the forest structure of Γ. A forest is a set of 1PI (one-particle-irreducible, i.e. the graph remains connected after cutting an arbitrary line) divergent subgraphs $\gamma \subset \Gamma$ which do not overlap. Instead, any two elements (= boxes) of a forest are either disjoint or nested. The forest structure is the collection of the maximal forests of Γ, i.e. the forests which are not contained in another forest. There are several maximal forests in general to a Feynman graph.

Kreimer defines [1] a recursive procedure to assign parenthesized words (PW) to the boxes of a maximal forest. The total graph Γ stands for a certain integrand I_Γ depending on external and internal momenta. A box is represented by a pair of opening-closing parentheses. Two nested boxes are represented by (( )) and two disjoint boxes by ( )( ). In an irreducible PW (iPW), the leftmost opening parenthesis matches its rightmost closing parenthesis. A primitive box contains no nested boxes and represents a graph γ without subdivergences. Examples of primitive boxes ( ) are (see (3) on top of the page). (The reader is encouraged to verify using (2) that the last three examples contain no divergent subgraphs.) We associate the integrand I_γ defined by the vertices and propagators of γ to such a primitive box and write the PW (I_γ). A non-primitive box contains nested boxes. It describes a graph γ with subdivergences γ_i, which are already characterized by PWs X_i. Examples for graphs with one nested subdivergence (( )) are (see (4) on top of the page).

Examples for graphs with two disjoint nested subdivergences: (( )( )) are (see (5) on top of the next page). And here are two examples for graphs with a nested subdivergence which has itself a nested subsubdivergence ((( ))):
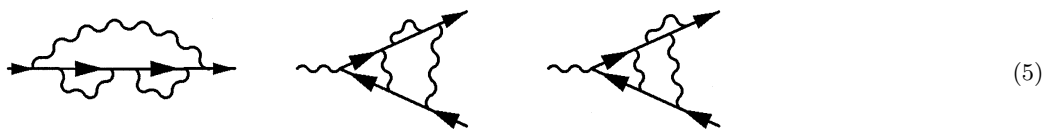

$$(6)$$

$$(5)$$

$$(((s_1)(v_2)v_4)(p_3)v_5) = \quad$$ $$(7)$$

If we shrink all nested boxes (=divergent subgraphs γ_i) of γ to points, there remains a fraction $I_{\gamma/\cup\gamma_i}$ of the integrand of γ defined by the vertices and propagators of $\gamma/\cup\gamma_i$. The latter should be regarded as a Feynman graph with holes at the places where the subgraphs γ_i had been before. We agree that for self-energy insertions γ_i splitting propagators into two, one of the new propagators belongs to the subgraph γ_i. In this way we keep the number of possible holes in a Feynman graph finite. We write the fraction $I_{\gamma/\cup\gamma_i}$ next to the right closing parenthesis and everything we have shrunk to a point (the X_i) between that fraction and the left opening parenthesis. The resulting PW looks like this: $\left(X_1 \ldots X_n \; I_{\gamma/\{\gamma_1\cup\ldots\cup\gamma_n\}}\right)$. Note that the order of disjoint boxes is irrelevant. For instance, the PW of the example (1) (considered as 1PI) looks as given in (7) (see (7) on top of the page).

A slash through a propagator means amputation and a small circle symbolizes a hole. We see that our building blocks are the Feynman graphs with possible holes at any vertex and in any propagator.

By this procedure we associate a PW to each maximal forest. As discovered by Kreimer [1], the PWs form a Hopf algebra whose antipode axiom reproduces the forest formula [7]. This assumes that overlapping divergences such as
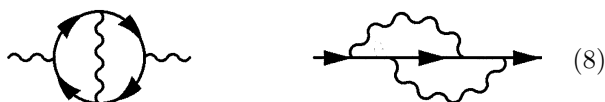
 $$(8)$$

have been disentangled into a linear combination of PWs containing disjoint and nested divergences exclusively, for instance via the Schwinger-Dyson equation, see [1,5]. The outcome is thus a linear combination of PWs each of them describing a maximal forest, and the forest formula is reduced to a rather trivial prescription.

The goal of this paper is to modify the PWs and the Hopf algebra operations in such a way that any 1PI-Feynman graph is described by *a single* PW and that all Hopf algebra operations are defined on such a PW. Our starting point is the observation that in the case of overlapping divergences there exist several maximal forests to a Feynman graph. It is clear that democracy requires to comprise all PWs associated to these maximal forests to one bigger object. We propose to build a column vector whose components are the PWs of maximal forests. The order of the components (or rows as they are long objects) of this vector is not relevant, of course. As the integrands associated to the PWs of each row are equal, we associate this universal integrand to our column vector.

There is one further modification necessary. Later on we are going to identify the subwords of such a vector and define the removal of subwords. Subwords represent subgraphs and the removal means replacing the subgraph by a hole. But subgraphs or subwords can occur identically in various maximal forests. If we now compare the maximal forests of a graph with removed subgraph and the maximal forests of the original graph, it is easy to see that the subgraph is removed in all maximal forests it had occurred. (An example is the step from (11) to (9) in the next section by cutting out loop 3.) We must implement this feature in our vectors. We propose to connect by a tree of lines the closing parentheses of identical and simultaneously shrinkable boxes. If we pull out a subword of such a vector and if the subword is connected over various rows, we simply have to remove all of them.

Thus, our PWs are vectors of one-line-PWs representing the maximal forests of a Feynman graph, where the closing parentheses of simultaneously shrinkable boxes are connected. We define now the notion of a parenthesized subword (PSW) of a PW. A PSW Y of X is everything between a set of connected closing parentheses and its matching opening parentheses. Disconnected rows of X which are accidentally between connected rows are not part of the PSW Y under consideration.

There is an algorithm which yields all PSW of a PW. Starting with the first row we run from the left through the PW until we meet a closing parenthesis. In general, it will be connected with other closing parentheses in different rows. These connected closing parentheses and their matching opening parentheses define our first PSW. We mark all these connected closing parentheses. We then go ahead and move through the first row until we arrive at the next closing parenthesis. This gives the next PSW and marks the next set of parentheses. We repeat this procedure until the rightmost closing parenthesis is reached. Then we pass to the second row and continue to search for new closing parentheses and related PSW, i.e. we ignore all parentheses marked in the previous steps. This search continues through all rows and stops at the lower right corner of our PW.

In what follows we will freely use the notions parenthesized word (PW), irreducible PW (iPW, the leftmost and rightmost parentheses match), primitive PW (no nested

divergences, a special iPW) and parenthesized subword (PSW, a special iPW). We remark that a possible extension could be the inclusion of superficially convergent 1PI-graphs ($d < 0$) with subdivergences. All finite integrands fuse and stand immediately before the rightmost closing parentheses.

We will give now some examples for Feynman graphs with overlapping divergences which are represented by parenthesized words of several maximal forests. The PSW of some of these examples are discussed and further evaluated in Sect. 7.

3 Examples for Feynman graphs with several maximal forests

In QED there is the following contribution to the photon propagator:

$$\left.\begin{matrix} ((v_1)p_2) \\ ((v_2)p_1) \end{matrix}\right] \qquad (9)$$

We can draw two maximal forests of boxes. We can first draw a box around the left loop which contains the vertex correction with interior momentum k_1. Then we put this box into the large box which encircles both loops. Or we can first enclose the right loop by a vertex box and then put everything into *the same* large box. Graphically, the two possibilities look like this:

$$= ((v_1)p_2) \qquad \text{or}$$

$$(10)$$

$$= ((v_2)p_1) \ .$$

In the first case, the innermost box is the primitive box (v_1) the integrand of which is – in the Feynman gauge – given by

$$v_{1A}^{\mu B} = \left[e\gamma^\kappa \frac{\not{k}_1+\mu}{k_1^2-\mu^2} e\gamma^\mu \frac{\not{k}_1+\not{p}+\mu}{(k_1+p)^2-\mu^2} e\gamma_\kappa \frac{1}{(k_1-k_2)^2-M^2} \right]_A^B$$

Here, e is the electron charge, μ is the electron mass and M an auxiliary photon mass to avoid IR-divergences. Capital roman letters label Clifford indices and greek letters

Lorentz indices. This vertex box is nested in the large box, so we must write $((v_1)p_2)$ as the maximal forest. The integrand p_2 is the interior of the large box after shrinking the small box (v_1) to a hole. What remains is loop 2 and the integrand is found to be

$$p_{2B}^{\nu A} = \left[\frac{\not{k}_2+\not{p}+\mu}{(k_2+p)^2-\mu^2} e\gamma^\nu \frac{\not{k}_2+\mu}{k_2^2-\mu^2} \right]_B^A$$

In the second case the loops 1 and 2 change their role and we obtain the maximal forest $((v_2)p_1)$ with

$$v_{2B}^{\nu A} = \left[e\gamma_\kappa \frac{\not{k}_2+\not{p}+\mu}{(k_2+p)^2-\mu^2} e\gamma^\nu \frac{\not{k}_2+\mu}{k_2^2-\mu^2} e\gamma^\kappa \frac{1}{(k_2-k_1)^2-M^2} \right]_B^A ,$$

$$p_{1A}^{\mu B} = \left[\frac{\not{k}_1+\mu}{k_1^2-\mu^2} e\gamma^\mu \frac{\not{k}_1+\not{p}+\mu}{(k_1+p)^2-\mu^2} \right]_A^B .$$

We have found two maximal forests $((v_1)p_2)$ and $((v_2)p_1)$ in this example. These two forests form the 2-line vector $\begin{pmatrix} ((v_1)p_2) \\ ((v_2)p_1) \end{pmatrix}$. However, the large box occurs identically in both maximal forests. We cannot shrink it in one of them and keep it in the other. Therefore, the closing parentheses representing the large box in both rows of the vector must be connected, as we have already indicated in (9).

Here is a graph with two maximal forests containing a nested divergence:

$$\left.\begin{matrix} (((v_3)v_{13})p_2) \\ (((v_3)v_{23})p_1) \end{matrix}\right] \qquad (11)$$

The vertex correction v_3 is nested in both vertex corrections v_{i3} comprising the common loop 3 and loop i. The subword (v_3) is identical in both maximal forests $(((v_3)v_{13})p_2)$ and $(((v_3)(v_{23})p_1)$. If we shrink it in one of them it is automatically removed in the other one. For the same reasons both maximal forests are connected at the outermost box.

Here is now a more complicated forest structure:

$$\left.\begin{matrix} ((v_1) (v_2) & p_3) \\ (((v_1) & v_{13})p_2) \\ ((& (v_2) v_{23})p_1) \end{matrix}\right] \qquad (12)$$

We have three possibilities for drawing disjoint boxes: We can take loops 1 and 2 and put them into the large box, or we can put loop 1 into the vertex box which covers loops 1 and 3 and then everything into the large box, or we can exchange the role of loops 1 and 2.

Let us also give an example from ϕ^4-theory. There is the following second-order correction to the propagator:

$$\begin{array}{r}((x_{23})y_1)\\((x_{31})y_2)\\((x_{12})y_3)\end{array} \qquad (13)$$

Here, x_{ij} is the vertex correction $\bowtie$ involving the lines i, j and y_k the tadpole graph $\multimap\!\!O\!\!\multimap$ involving the line k. The three maximal forests are connected because shrinking one of them to a hole forces the reduction of the other two.

4 Kreimer's R-operation [1]

To any PW X, Kreimer associates a second, in a certain sense equivalent copy $R[X]$. The philosophy is that $R[X]$ is a local counterterm, a point-like interaction. It is so to say a new vertex, mass or kinetic term in the Lagrangian, which itself is infinite but such that a certain combination of counterterms and divergent 1PI graphs is finite. The finite linear combination in question is given by the forest formula or – as discovered by Kreimer – by the antipode axiom of a (quasi-) Hopf algebra to construct. For renormalizability it is essential that all counterterms can be absorbed by a redefinition of physical parameters of the theory. In particular in gauge theories there are potentially more types of counterterms than physical parameters [6]. It is important then that counterterms and divergences of the sum of all graphs contributing to a certain amplitude cancel. We avoid a discussion of these subtleties by considering scalar theories or – with some care – QED.

The R-operation depends on the renormalization scheme, which in principle is arbitrary but fixed throughout the investigation. We shall work in the BPHZ scheme [8,9,7] which is the standard one in connection with the forest formula. A iPW X represents one box containing a divergent Feynman graph with in general several forests of subdivergences. The box has n_B bosonic and n_F fermionic external legs. The superficial degree of divergence $d[X]$ of the iPW X is bounded by the power counting theorem (2), $d[X] \leq 4-n_B-\frac{3}{2}n_F$. In the BPHZ scheme the integrand $R[X]$ is the Taylor expansion until order $d[X]$ with respect to the external momenta of X. We call $X = \prod_i X_i$ a tree if each $X_j \subset X$ has a common momentum variable with at least one $X_i \subset X$, $i \neq j$. In this case we define $R[X]$ to be the Taylor expansion with respect to the external momenta of the smallest possible iPW $\tilde{X}$ containing all X_i as subwords. Finally, for X being a product of disjoint trees X_t, we define $R[\prod X_t] = \prod R[X_t]$. Note that in general $X - R[X]$ is an integrand yielding a finite integral only if X is a primitive PW without subdivergences.

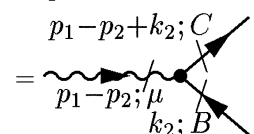To give an example, consider the divergent Feynman graph with subdivergence given in (14).

$$= \left(\left(\raisebox{-1em}{\includegraphics{}} \right) \raisebox{-1em}{\includegraphics{}} \right) = ((v_1)v_2)\ , \qquad (14)$$

$$v_{1B}^{\mu C} = \Big[e\gamma^\nu\, \tfrac{\slashed{k}_1+\mu}{k_1^2-\mu^2}\, e\gamma^\mu\, \tfrac{\slashed{k}_1+(\slashed{p}_1-\slashed{p}_2+\slashed{k}_2)-\slashed{k}_2+\mu}{(k_1+(p_1-p_2+k_2)-k_2)^2-\mu^2}$$
$$\times e\gamma_\nu\, \tfrac{1}{(k_1-k_2)^2-M^2}\Big]_B^C\ ,$$

$$v_{2AC}^{BD} = \Big[e\gamma^\kappa\, \tfrac{\slashed{k}_2+\mu}{k_2^2-\mu^2}\Big]_A^B$$
$$\times \Big[\tfrac{\slashed{p}_1-\slashed{p}_2+\slashed{k}_2+\mu}{(p_1-p_2+k)^2-\mu^2}\, e\gamma_\kappa\, \tfrac{1}{(k_2-p_2)^2-M^2}\Big]_C^D\ .$$

We have written v_1 in a form where its external momenta $p_1-p_2+k_2$ and k_2 are explicit. The two subwords of $((v_1)v_2)$ are clearly (v_1) and $((v_1)v_2)$. Let us compute $R[(v_1)]$. It has 2 fermionic and 1 bosonic external legs, hence $d[(v_1)] \leq 0$, and actually $d[(v_1)] = 0$. In the BPHZ scheme we take the Taylor expansion of (v_1) in its external momenta $p_1-p_2+k_2$ and k_2 until order 0. This gives
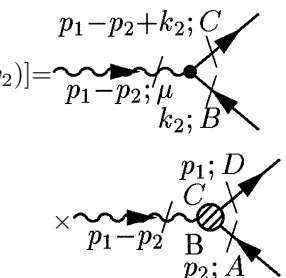
$$R[(v_1)] = v_{1B}^{\mu C}\big|_{p_1-p_2+k_2=k_2=0}$$
$$= \Big[e\gamma^\nu\, \tfrac{\slashed{k}_1+\mu}{k_1^2-\mu^2}\, e\gamma^\mu\, \tfrac{\slashed{k}_1+\mu}{k_1^2-\mu^2}\, e\gamma_\nu\, \tfrac{1}{k_1^2-M^2}\Big]_B^C$$

$$(15)$$

We see that $R[(v_1)]$ defines a local counterterm, and the integral $\int d^4k_1\, \{(v_1) - R[(v_1)]\}$ is finite.

We can now apply the R-operation to the PWs $((v_1)v_2)$ and $R[(v_1)](v_2)$. In both cases this means Taylor expansion with respect to the external momenta p_1, p_2 of $((v_1)v_2)$ until degree $d[((v_1)v_2)] = 0$, because $R[(v_1)]$ and (v_2) have common momenta p_1, p_2, k_2. We obtain

$$R[R[(v_1)](v_2)] = \raisebox{-1em}{\includegraphics{}}$$

$$(16a)$$

$$= \left[e\gamma^\nu \, \tfrac{\not{k}_1+\mu}{k_1^2-\mu^2} \, e\gamma^\mu \, \tfrac{\not{k}_1+\mu}{k_1^2-\mu^2} \, e\gamma_\nu \, \tfrac{1}{k_1^2-M^2} \right]_B^C$$

$$\times \left[e\gamma^\kappa \, \tfrac{\not{k}_2+\mu}{k_2^2-\mu^2} \right]_A^B \, \left[\tfrac{\not{k}_2+\mu}{k_2^2-\mu^2} \, e\gamma_\kappa \, \tfrac{1}{k_2^2-M^2} \right]_C^D \, ,$$

$$R[((v_1)v_2)] = \text{(figure)}$$

$$= \left[e\gamma^\kappa \, \tfrac{\not{k}_2+\mu}{k_2^2-\mu^2} \, e\gamma^\nu \, \tfrac{\not{k}_1+\mu}{k_1^2-\mu^2} \, e\gamma^\mu \, \tfrac{\not{k}_1+\mu}{k_1^2-\mu^2} \right.$$

$$\times e\gamma_\nu \, \tfrac{1}{(k_1-k_2)^2-M^2} \, \times$$

$$\left. \times \, \tfrac{\not{k}_2+\mu}{k^2-\mu^2} \, e\gamma_\kappa \, \tfrac{1}{k_2^2-M^2} \right]_A^D \, . \quad (16b)$$

Both $R[R[(v_1)](v_2)]$ and $R[((v_1)v_2)]$ define local counterterms, but both integrals $\int d^4k_2 d^4k_1 \quad \{((v_1)v_2) -R[((v_1)v_2)]\}$ and $\int d^4k_2 d^4k_1 \, \{((v_1)v_2) - R[R[(v_1)](v_2)]\}$ are *infinite*. To obtain a finite expression one has to include $R[(v_1)](v_2)$ in a way given by the forest formula.

We must say a few words how equivalence is defined quantitatively. Renormalization schemes depend on some regularization parameter ϵ. Infinities correspond to pole terms in ϵ. In terms of ϵ, Kreimer gives the following definition of equivalence:

$$X \sim Y \qquad \text{iff} \qquad \lim_{\hbar \to 0, \epsilon \to 0} \{X - Y\} = 0 \, . \quad (17)$$

Accordingly, R is a renormalization map iff $R[X] \sim X$ for all PWs X. It is important to understand that $R[X] \sim X$ does not imply $R[X]Y \sim XY$. The reason is that if Y has pole terms in ϵ then in the product $(R[X] - X)Y$ also terms of order ϵ in $R[X] - Y$ become essential. It turns out that the full set of properties of a Hopf algebra can only be guaranteed if equivalence works for products, in a certain sense. The precise condition to the the renormalization map R is

$$R\Big[\prod_i R[X_i] \prod_j Y_j \Big] = \prod_i R[X_i] \prod_j R[Y_j] \, . \quad (18)$$

We indicate by $X \approx Y$ that under the condition (18) we have $X \sim Y$, but that in general equivalence is not guaranteed.

In the BPHZ scheme there is no regularization parameter ϵ, so we cannot use the definition (17). Nevertheless, R is defined for any Feynman graph, and we say that $X \sim Y$ iff $Y = X$ or $Y = R[X]$. The condition (18) makes sense, and we have $R^2 = R$ by construction. We remark that superficially convergent graphs with subdivergences (if included, see the remark at the end of Sect. 2) are annihilated by R. This is clear in the BPHZ scheme, because a Taylor expansion until order $d < 0$ makes no sense. In what follows we work on a general level without specifying the renormalization scheme and its R-operation.

5 The Hopf algebra

Following the work of Kreimer [1] we will now equip the PWs with the structure of a (quasi-) Hopf algebra. This goes in four steps. First, we would like to consider the set $\mathcal{A}$ of all PWs (which include from now on its R-equivalents) as a vector space. We enlarge formally this set $\mathcal{A}$ by all rational linear combinations of PWs. This makes $\mathcal{A}$ to a formal vector space over the field $\mathbf{Q}$ of rational numbers, $\mathbf{Q}$ just for simplicity.

The second step makes $\mathcal{A}$ to an algebra by defining a product m. This is an operation which assigns to a sum of pairs of elements of $\mathcal{A}$ a new one. Actually only $\mathbf{Q}$-equivalence classes of pairs are essential so that m operates on the tensor product, $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. According to [1] we build the commutative and associative formal product

$$m[X \otimes Y] = XY = YX \, , \qquad X, Y \in \mathcal{A} \, ,$$

corresponding to two disjoint divergences. We further define a formal unit e by

$$m[e \otimes X] = m[X \otimes e] = X \qquad \forall \, X \in \mathcal{A} \, .$$

The unit e is *not* considered as a PSW. It is convenient to consider e as produced by an operation

$$E : \mathbf{Q} \to \mathcal{A} \, , \qquad E(q) = qe \, .$$

The third step is to make $\mathcal{A}$ to a coalgebra. The operations of a coalgebra are the duals of the algebra operations. Dual means turning the arrows. For instance, the dual of the above unit E, the counit ε, will be a formal operation given by

$$\varepsilon : \mathcal{A} \to \mathbf{Q} \, ,$$
$$\varepsilon[qe] := q \, , \qquad \varepsilon[X] := 0 \, \forall X \neq e \, , \ X \in \mathcal{A} \, .$$

Now comes a physically significant ingredient of our coalgebra, the coproduct Δ. A product was the assignment of one element to sums of pairs of other elements. Hence, a coproduct will be the splitting of one element into sums of pairs of other elements, in symbols

$$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \, .$$

The philosophy is that Δ provides the splitting of a 1PI-graph Γ into a formal sum of tensor products of all possible divergent subgraphs γ_i (left factor) by the fraction Γ/γ_i obtained by reducing γ_i to a hole (right factor). The left factors are, moreover, treated by the R-operation.

Let us formalize this idea. The graph Γ is represented by a PW X describing its forest structure. Let $\{X_i\}$ be a subset of PSWs of X in the sense of Sect. 2. We are going to define the fraction $X/\prod_i X_i$. If $\prod_i X_i = X$ we define $X/X = e$. Otherwise we label the rows of X. Each row of X_i is a substring of one determined row of X. We give to the X_i-rows the labels of the X-rows they are contained in. These labels could be ambiguous but we fix one choice for all subwords of X. We delete from X and all X_i all but those rows whose labels occur in each of the chosen

PSWs X_i. Let the results be X' and X_i'. If there remains no row at all or if $X_i' \cap X_j' \neq \emptyset$ for some pair $\{X_i', X_j'\}$ then we put $X/\prod_i X_i = \emptyset$. Otherwise $X/\prod X_i$ is given by removing all X_i' from X'.

Now, the coproduct of a PW X containing the PSWs $X_1, \ldots X_n$ is defined by

$$\Delta[e] := e \otimes e \,,$$
$$\Delta[X] := e \otimes X + \sum_T \left\{ \prod_{i \in T} R[X_i'] \otimes X/\prod_{i \in T} X_i \right\} , \quad (19)$$

where the sum runs over all ordered subsets $T = \{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$, $i_1 < i_2 < \ldots < i_k$. The order of the factors and products is not important in this definition, but we must avoid taking identical terms several times. In the sequel we will omit the primes on X_i' which indicate the truncation to the common rows.

Our algebra $\mathcal{A}$ also contains elements of the type $R[X]$, where X is a PW. Kreimer gives two possible definitions for $\Delta \circ R$,

$$\Delta[R[X]] = \Delta[X] \,, \quad (20a)$$
$$\Delta[R[X]] = (\mathrm{id} \otimes R)' \circ \Delta[X] \,, \quad (20b)$$

where the prime means that $R[e]$ is replaced by e. Kreimer chooses to work with (20a). This choice violates coassociativity, but non-coassociativity is interesting from a number theoretical point of view [5]. We prefer (20b), because $R[X]$ is always a local counterterm $\bullet$. The philosophy is that Δ splits a graph into subgraphs and reminders. Hence, both of them should be local counterterms in this example, $\Delta[\bullet] = \sum \bullet \otimes \bullet$, and for us the natural definition is (20b) or

$$\Delta[R[X]] := e \otimes R[X]$$
$$+ \sum_T \left\{ \prod_{i \in T} R[X_i] \otimes R[X/\prod_{i \in T} X_i]' \right\} . \quad (21)$$

Again, the prime means that $R[X/X]$ has to be replaced by e instead of $R[e]$. This can be easily interpreted in terms of PSWs. The PSWs X_i of $R[X]$ are identical with the PSWs of X, except for the total PW $R[X]$. The fraction $R[X]/\prod_i X_i$ obtained by removing the PSWs X_i in $R[X]$ clearly coincides with $R[X/\prod_i X_i]$, except for $R[X]/R[X] = e$.

There are of course some consistency conditions to fulfill before we can call $\mathcal{A}$ a coalgebra. One of these conditions to Δ is coassociativity, which is derived from associativity by turning the arrows: If we split one element into a sum of pairs, it must be the same to split the left or the right factor further. In symbols, coassociativity means

$$(\mathrm{id} \otimes \Delta) \circ \Delta[X] = (\Delta \otimes \mathrm{id}) \circ \Delta[X] \,, \quad \forall X \in \mathcal{A} \,. \quad (22)$$

We give the proof in proposition 1 in the appendix. For the choice (20a), coassociativity was only satisfied under the additional condition (18), but also with (20b) we need (18) to get a true Hopf algebra, see below.

The 'counit' ε is only a left counit and becomes a true counit under the condition (18). Recall that an element of $\mathcal{A}$ is a formal linear combination of products $X = \prod_i X_i \ \prod_j R[Y_j]$, where X_i, Y_j are iPWs. We have for $X \neq e$

$$\Delta[X] = \prod_i R[X_i] \ \prod_j R[Y_j] \otimes e$$
$$+ e \otimes \prod_i X_i \ \prod_j R[Y_j] + \sum Z \otimes Z' \,,$$

where Z, Z' stand for terms which do not contain the unit e and which are annihilated by ε. Hence, the counit axioms read

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta[X] = \prod_i X_i \ \prod_j R[Y_j] = X \,, \quad (23a)$$

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta[X] = \prod_i R[X_i] \ \prod_j R[Y_j] \approx R[X] \sim X. \quad (23b)$$

In the last line we need (18) to obtain equivalence with X. Moreover, the 'antipode' S defined below turns out to require (18) to be a true antipode.

So far we have equipped $\mathcal{A}$ with the structures of an algebra and a coalgebra. Both merge to a bialgebra if Δ is an algebra homomorphism,

$$\Delta \circ m[X \otimes Y] = (m \otimes m) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})[\Delta[X] \otimes \Delta[Y]] \,. \quad (24)$$

Here, $\tau[X \otimes Y] := Y \otimes X$ denotes the flip operation. It is evident that (24) is fulfilled, because the subwords of XY are the subwords X_i of X and Y_i of Y together.

The last step extends the bialgebra to a Hopf algebra. On a Hopf algebra there exists the additional structure of an antipode $S : \mathcal{A} \to \mathcal{A}$, which is the dual of the inverse in an algebra. Our algebra does not have an inverse (except for $e^{-1} = e$), nevertheless it has (under the condition (18)) an antipode, which will provide the link to the forest formula:

$$S[e] = e \,, \quad (25a)$$
$$S[XY] = S[Y]S[X] \,, \qquad \forall X, Y \in \mathcal{A} \,, \quad (25b)$$
$$S[X] = -X - m \circ (\mathrm{id} \otimes S) \circ P_2 \circ \Delta[X] \,,$$
$$\forall \text{ iPW } X \in \mathcal{A} \,, \quad (25c)$$
$$S[R[X]] = -R[X + m \circ (S \otimes \mathrm{id}) \circ P_2 \circ \Delta[X]] \,,$$
$$\forall \text{ iPW } X \in \mathcal{A} \,, \quad (25d)$$

where $P_2 = (\mathrm{id} - E \circ \varepsilon) \otimes (\mathrm{id} - E \circ \varepsilon)$. The antipode is by (25a) recursively defined, because in $P_2 \circ \Delta[X]$ only smaller words than X survive, and for primitive words (x) we simply have $S[(x)] = -(x)$ and $S[R[(x)]] = -R[(x)]$. We show in proposition 2 that of the four axioms on S to check, only one is fulfilled in general renormalization schemes, the other three require (18):

$$m \circ (S \otimes \mathrm{id}) \circ \Delta[X] \sim E \circ \varepsilon[X] \,, \quad (26a)$$
$$m \circ (\mathrm{id} \otimes S) \circ \Delta[X] \approx E \circ \varepsilon[X] \,, \quad (26b)$$
$$m \circ (S \otimes \mathrm{id}) \circ \Delta[R[X]] \approx E \circ \varepsilon[R[X]]$$
$$\approx m \circ (\mathrm{id} \otimes S) \circ \Delta[R[X]] \,. \quad (26c)$$

Formula (26a) relies deeply on the fact that for X being an iPW, the equation

$$
\begin{aligned}
&m \circ (S \otimes \mathrm{id}) \circ \Delta[X] \\
&= (\mathrm{id}-R)\Big[X + \sum_T \Big\{ m\Big[\prod_{i \in T}(-R[\bar{X}_i]) \otimes X/\prod_{i \in T} X_i\Big]\Big\}\Big] \\
&= (\mathrm{id}-R)[\bar{X}]\,, \\
&R[\bar{X}_i] := -S[R[X_i]]\,,
\end{aligned}
\tag{27a}
$$

reproduces Bogoliubov's recurrence formula of renormalization [10]. Here, $X_i \neq X$, $i = 1,\dots,n$, are the proper PSWs of X. Denoting by $X_{ij} \neq X_i$, $j = 1,\dots n_i$, the proper PSW of X_i, we can write

$$
\begin{aligned}
R[\bar{X}_i] &\equiv -S[R[X_i]] \\
&= R[X_i + m \circ (S \otimes \mathrm{id}) \circ P_2 \circ \Delta[X_i]] \\
&= R\Big[X_i + \sum_{T_i} \Big\{ m\Big[\prod_{j \in T_i} S[R[X_{ij}]] \otimes X_i/\prod_{j \in T_i} X_{ij}\Big]\Big\}\Big]\,.
\end{aligned}
\tag{27b}
$$

Thus, $\bar{X}_i$ has the same structure as $\bar{X}$, and we obtain indeed a recurrence formula. The integrand $\bar{X}$ associated to an integrand X is pre-finite, which means that all subdivergences are compensated. The remaining superficial divergence is compensated by $\mathrm{id}-R$.

To identify (27a) with Bogoliubov's recurrence formula it is important that the coproduct produces all combinations of disjoint subdivergences, which are encoded in the set of maximal forests. This means that in describing a Feynman graph Γ with subdivergences by a parenthesized word X, we must somehow include in X all maximal forests of Γ. That is why we have written the maximal forests as lines of X. The maximal forests are defined by the relative position of the subdivergences. Each time we meet an overlap of subdivergences we have a branching of forests. Having defined the forests we must say how to detect the disjoint subdivergences. Forests contain by definition no overlapping divergences, so the only problem is to avoid nested divergences. This was achieved by our factorization procedure $X/\prod_{i \in T} X_i$, which yields zero if the X_i intersect. By variation of T (which must be an ordered set to avoid the multiplicities) we get all products of disjoint subdivergences. It is important that if a subdivergence occurs in two or more forests, we must count it only once. That is why we have introduced the brackets connecting identical regions in various maximal forests.

In conclusion, our modified definition of a parenthesized word that keeps track of different maximal forests and connects simultaneously shrinkable boxes is the correct language for Bogoliubov's recurrence formula [10]. This formula has an explicit solution, Zimmermann's forest formula [7]. Both are reproduced by coproduct and antipode of a (quasi-) Hopf algebra via $m \circ (S \otimes \mathrm{id}) \circ \Delta$. We remark that the crucial formula (26a) is actually a stronger equivalence $\simeq$. Due to the forest formula (27a), the difference between left and right hand sides is *finite* in any renormalization scheme.

6 The primitivator $\mathcal{P}$ and the relation to the Hopf algebra of Kreimer

Having worked out a Hopf algebra of Feynman graphs where overlapping divergences are treated on the same footing as disjoint and nested ones, we must also say what the precise relation is to Kreimer's formulation [1] where overlapping divergences are resolved before building the Hopf algebra. Our presentation is inspired by an idea of Dirk Kreimer. A detailed discussion of these questions based on set-theoretical considerations was given in [4], some remarks can be found in the appendix of [2].

The connection to Kreimer's Hopf algebra is achieved by introduction of a "primitivator" $\mathcal{P}$ which maps overlapping divergences to primitive elements. Let X be an iPW with proper PSWs $X_i \neq X$, $i = 1,\dots,n$, and $T \subset \{1,\dots,n\}$. Let us write the outermost parentheses of iPWs explicitly, i.e. (X) instead of X and (X_i) instead of X_i and $(\mathcal{P}[X/\prod_{i \in T} X_i])$ instead of $\mathcal{P}[X/\prod_{i \in T} X_i]$. With this convention we define

$$
\mathcal{P}[(X)] := (X) - \sum_T \Big(\prod_{i \in T}(X_i)\,\mathcal{P}\big[X/\prod_{i \in T} X_i\big]\Big)\,.
\tag{28}
$$

We are going to prove that $\mathcal{P}[(X)]$ is primitive in the following sense:

$$
\Delta[\mathcal{P}[(X)]] = e \otimes \mathcal{P}[(X)] + R[\mathcal{P}[(X)]] \otimes e\,.
\tag{29}
$$

If (X) is primitive it contains no PSWs. Hence we have $T = \emptyset$ and $\mathcal{P}[(X)] = (X)$. For (X) and (Y) being primitive we compute $\mathcal{P}[((Y)X)] = ((Y)X) - ((Y)X) = 0$. By induction it is easy to show that $\mathcal{P}[Y] = 0$ for any non-primitive one-line iPW Y. To prove (29) by induction we assume that all $(\mathcal{P}[X/\prod_{i \in T} X_i])$ are primitive in the sense (29). Hence the only PSWs of $\Big(\prod_{i \in T}(X_i)\mathcal{P}\big[X/\prod_{i \in T} X_i\big]\Big)$ are the (X_i) and their subwords (X_{k_i}), with $k_i \in T^i \subset \{1,\dots n_i\}$. We compute

$$
\begin{aligned}
&\Delta[\mathcal{P}[(X)]] = e \otimes (X) + R[(X)] \otimes e \\
&+ \sum_T \prod_{i \in T} R[(X_i)] \otimes (X/\prod_{i \in T} X_i) \\
&- \sum_T \Big\{ e \otimes \Big(\prod_{i \in T}(X_i)\,\mathcal{P}\big[X/\prod_{i \in T} X_i\big]\Big) \\
&+ R\Big[\Big(\prod_{i \in T}(X_i)\,\mathcal{P}\big[X/\prod_{i \in T} X_i\big]\Big)\Big] \otimes e \\
&+ \prod_{i \in T} R[(X_i)] \otimes \big(\mathcal{P}[X/\prod_{i \in T} X_i]\big)\Big\} \\
&- \sum_{T_1,T_2,T_3,\bigcup_{m \in T_2} T^m} \Big\{ \prod_{i \in T_3} R[(X_i)] \prod_{m \in T_2}\Big\{\prod_{k_m \in T^m} R[(X_{k_m})]\Big\} \otimes \\
&\otimes \Big(\prod_{l \in T_1}(X_l) \prod_{m \in T_2}(X_m/\prod_{k_m \in T^m} X_{k_m})\,\mathcal{P}\big[X/\prod_{j \in T_1 \oplus T_2 \oplus T_3} X_j\big]\Big)\Big\}.
\end{aligned}
\tag{30}
$$

In the last (splitted) line we have $T_1 \oplus T_2 \neq \emptyset$ because that contribution has been written explicitly in the line

before. Comparison of (A.6) with (A.4) in the appendix shows that the last (splitted) line of (30) equals

$$-\sum_{T,T'}\Big\{\prod_{i\in T}R[(X_i)]\otimes$$
$$\otimes\Big(\prod_{j\in T'}(\{X/\prod_{i\in T}X_i\}_j)\,\mathcal{P}[\{X/\prod_{i\in T}X_i\}/\{X/\prod_{i\in T}X_i\}_j]\Big)\Big\},$$

where $\{X/\prod_{i\in T}X_i\}_j$, $j\in T'$, are the PSWs of $(X/\prod_{i\in T}X_i)$. Using the definition (28) for (X) and $(X/\prod_{i\in T}X_i)$ we confirm (29).

This means that we may replace the overlapping divergence (X) by the linear combination $\mathcal{P}[(X)]+\sum_T\Big(\prod_{i\in T}(X_i)\mathcal{P}\big[X/\prod_{i\in T}X_i\big]\Big)$. If (X) is an overlapping divergence which contains no overlapping subdivergences, all X_i are one-line PWs (or connected identical rows of one-line PWs $\tilde{X}_i$; in that case we replace X_i by $\tilde{X}_i$) . Since the $\mathcal{P}[(X)]$ form additional primitive (i.e. one-line) elements of the Hopf algebra, we have written the multi-line overlapping divergence (X) as a linear combination of one-line PWs. In other words, our Hopf algebra is isomorphic to a Hopf algebra of one-line PWs, and this is precisely Kreimer's original Hopf algebra. The primitive elements of Kreimer's Hopf algebra are the graphically primitive elements and from each overlapping divergence a computational-primitive element. Our approach provides an explicit construction of the latter. The same can be achieved, for instance, by Schwinger-Dyson techniques [1, 5] or set-theoretical considerations [4].

The advantage of Kreimer's Hopf algebra of one-line PWs is that it can be reformulated as a Hopf algebra of rooted trees [2]. A subalgebra thereof turns out to be the dual of the diffeomorphism group of a manifold. It is now interesting to ask [2] for the (noncommutative) manifold whose diffeomorphism group is the dual of the Hopf algebra of renormalization. We feel that answering this question is indispensable for a true understanding of renormalization and of the short-distance structure of spacetime.

7 Two examples for the coproduct and the forest formula

We compute here the coproducts and forest formulas for two striking examples of Sect. 3. By PSW we shall always mean proper PSW, we write the trivial PWs explicitly. The proper PSWs of

$$X=\begin{smallmatrix}((v_1)p_2]\\((v_2)p_1]\end{smallmatrix}\qquad \text{(9)}$$

are obviously

$$X_1=(v_1)\,,\qquad X_2=(v_2)\,.\qquad\qquad(9_{\mathrm{s}})$$

Let us compute X/X_1. The only row of X_1 can only be related to the upper row of X so that $X'=((v_1)p_2)$. To

obtain X/X_1 we must remove X_1 from X', the result is $X/X_1=(p_2)$. Accordingly,

$$X/X_1=(p_2)\,,\quad X/X_2=(p_1)\,,\quad X/(X_1X_2)=0\,.\qquad(9_{\mathrm{r}})$$

The last equation holds because X_1,X_2 have no common row label. Therefore, the coproduct reads

$$\Delta[X]=e\otimes\begin{smallmatrix}((v_1)p_2]\\((v_2)p_1]\end{smallmatrix}+R\Big[\begin{smallmatrix}((v_1)p_2]\\((v_2)p_1]\end{smallmatrix}\Big]\otimes e$$
$$+R[(v_1)]\otimes(p_2)+R[(v_2)]\otimes(p_1)\,.\qquad(9_\Delta)$$

Let us now apply the operator $m\circ(S\otimes\mathrm{id})$. To avoid unnecessary calculation we use the general result (27a),

$$m\circ(S\otimes\mathrm{id})\circ\Delta[X]$$
$$=(\mathrm{id}-R)[X+m\circ(S\otimes\mathrm{id})\circ P_2\circ\Delta[X]]\,.$$

The projection P_2 removes all terms containing the unit e so that in the case under consideration we have $P_2\circ\Delta[X]]=R[(v_1)]\otimes(p_2)+R[(v_2)]\otimes(p_1)$. This gives

$$m\circ(S\otimes\mathrm{id})\circ\Delta[X]$$
$$=(\mathrm{id}-R)\left[\begin{smallmatrix}((v_1)p_2]\\((v_2)p_1]\end{smallmatrix}+S[R[(v_1)]](p_2)+S[R[(v_2)]](p_1)\right]$$
$$=(\mathrm{id}-R)\left[\begin{smallmatrix}((v_1)p_2]\\((v_2)p_1]\end{smallmatrix}-R[(v_1)](p_2)-R[(v_2)](p_1)\right]\,.\qquad(9_{\mathrm{f}})$$

The primitivator of X reads

$$o_1:=\mathcal{P}[X]=\begin{smallmatrix}((v_1)p_2]\\((v_2)p_1]\end{smallmatrix}-((v_1)p_2)-((v_2)p_1)\,.\qquad(9_{\mathrm{p}})$$

It is easy to verify $\Delta[o_1]=R[o_1]\otimes e+e\otimes o_1$.

Let us repeat the same steps for example (11):

$$X=\begin{smallmatrix}(((v_3]v_{13})p_2]\\(((v_3]v_{23})p_1]\end{smallmatrix}\,,\qquad\qquad\text{(11)}$$

$$X_1=\begin{smallmatrix}(v_3]\\(v_3]\end{smallmatrix}\,,\quad X_2=((v_3)v_{13})\,,\quad X_3=((v_3)v_{23})\,,\qquad(11_{\mathrm{s}})$$

$$X/X_1=\begin{smallmatrix}((v_{13})p_2]\\((v_{23})p_1]\end{smallmatrix}\,,\quad X/X_2=(p_2)\,,\qquad X/X_3=(p_1)\,,$$
$$X/(X_1X_2)=X/(X_1X_3)=X/(X_2X_3)=0\,,$$
$$X/(X_1X_2X_3)=0\,,\qquad\qquad(11_{\mathrm{r}})$$

$$\Delta[X]=e\otimes\begin{smallmatrix}(((v_3]v_{13})p_2]\\(((v_3]v_{23})p_1]\end{smallmatrix}+R\left[\begin{smallmatrix}(((v_3]v_{13})p_2]\\(((v_3]v_{23})p_1]\end{smallmatrix}\right]\otimes e$$
$$+R[(v_3)]\otimes\begin{smallmatrix}((v_{13})p_2]\\((v_{23})p_1]\end{smallmatrix}$$
$$+R[((v_3)v_{13})]\otimes(p_2)+R[((v_3)v_{23})]\otimes(p_1)\,,\qquad(11_\Delta)$$

(in the third term, $\begin{smallmatrix}(v_3]\\(v_3]\end{smallmatrix}$ can be condensed to (v_3))

$$m\circ(S\otimes\mathrm{id})\otimes\Delta[X]$$
$$=(\mathrm{id}-R)\left[\begin{smallmatrix}(((v_3]v_{13})p_2]\\(((v_3]v_{23})p_1]\end{smallmatrix}+S[R[(v_3)]]\begin{smallmatrix}((v_{13})p_2]\\((v_{23})p_1]\end{smallmatrix}\right.$$

$$+ S\big[R[((v_3)v_{13})]\big](p_2) + S\big[R[((v_3)v_{23})]\big](p_1)\bigg]$$

$$= (\mathrm{id} - R)\Bigg[\genfrac{}{}{0pt}{}{(((v_3)}{(((v_3)}\genfrac{}{}{0pt}{}{v_{13})p_2)}{v_{23})p_1)} - R[(v_3)]\genfrac{}{}{0pt}{}{((v_{13})p_2)}{((v_{23})p_1)}$$

$$- \Big\{R[((v_3)v_{13})]$$

$$+ R\big[m \circ (S \otimes \mathrm{id}) \circ P_2 \Delta[((v_3)v_{13})]\big]\Big\}(p_2)$$

$$- \Big\{R[((v_3)v_{23})]$$

$$+ R\big[m \circ (S \otimes \mathrm{id}) \circ P_2 \Delta[((v_3)v_{23})]\big]\Big\}(p_1)\Bigg]$$

$$= (\mathrm{id} - R)\Bigg[\genfrac{}{}{0pt}{}{(((v_3)}{(((v_3)}\genfrac{}{}{0pt}{}{v_{13})p_2)}{v_{23})p_1)} - R[(v_3)]\genfrac{}{}{0pt}{}{(v_{13})p_2)}{(v_{23})p_1)}$$

$$- R[((v_3)v_{13})](p_2) + R\big[R[(v_3)](v_{13})\big](p_2)$$

$$- R[((v_3)v_{23})](p_1) + R\big[R[(v_3)](v_{23})\big](p_1)\Bigg], \quad (11_\mathrm{f})$$

$$o_2 := \mathcal{P}[X] = \genfrac{}{}{0pt}{}{(((v_3)}{(((v_3)}\genfrac{}{}{0pt}{}{v_{13})p_2)}{v_{23})p_1)} - ((v_3)o_1)$$

$$- (((v_3)v_{13})p_2) - (((v_3)v_{23})p_1). \quad (11_\mathrm{p})$$

The primitive element o_1 computed in (9_p) enters the decomposition of X into one-line PWs.

    Example (12) is similar to (11) and is left as an exercise to the reader. Example (13) is the obvious generalization of (9) to three maximal forests.

Appendix
Verification of the Hopf algebra properties

Proposition 1 *The coproduct Δ is coassociative, i.e.*
$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$.

Proof. Let X be an iPW which is not $R[X']$. Let $X_i \neq X$, $i = 1, \ldots, n$, be the proper PSW of X. Let T be the set of all ordered subsets of $\{1, 2, \ldots, n\}$. We write the contribution of the trivial PSW X of X explicitly:

$$\Delta[X] = e \otimes X + R[X] \otimes e + \sum_T \Big\{\prod_{i \in T} R[X_i] \otimes X/\prod_{i \in T} X_i\Big\}.$$

This gives

$$(\mathrm{id} \otimes \Delta) \circ \Delta[X] \quad\quad (A.1)$$

$$= e \otimes \Big\{e \otimes X + R[X] \otimes e + \sum_T \Big\{\prod_{i \in T} R[X_i] \otimes X/\prod_{i \in T} X_i\Big\}\Big\}$$

$$+ R[X] \otimes e \otimes e$$

$$+ \sum_T \Big\{\prod_{i \in T} R[X_i] \otimes e \otimes X/\prod_{i \in T} X_i\Big\}$$

$$+ \sum_T \Big\{\prod_{i \in T} R[X_i] \otimes R[X/\prod_{i \in T} X_i] \otimes e\Big\}$$

$$+ \sum_T \Big\{\prod_{i \in T} R[X_i] \otimes \sum_{T'} \Big\{\prod_{j \in T'} R[\{X/\prod_{i \in T} X_i\}_j] \otimes$$

$$\otimes \{X/\prod_{i \in T} X_i\}/\prod_{j \in T'} \{X/\prod_{i \in T} X_i\}_j\Big\}\Big\},$$

where $\{X/\prod_{i \in T} X_i\}_j$ are the proper PSW of $X/\prod_{i \in T} X_i$, $j = 1, \ldots, n' < n$, and T' is the set of all ordered subsets of $\{1, \ldots, n'\}$. The following terms can be rearranged:

$$e \otimes e \otimes X + \Big\{e \otimes R[X] \otimes e + R[X] \otimes e \otimes e$$

$$+ \sum_T \Big\{\prod_{i \in T} R[X_i] \otimes R[X/\prod_{i \in T} X_i]\Big\} \otimes e\Big\}$$

$$= (\Delta \otimes \mathrm{id})(e \otimes X + R[X] \otimes e) \quad\quad (A.2)$$

so that there remain

$$\sum_T \Big\{\prod_{i \in T} R[X_i] \otimes e \otimes X/\prod_{i \in T} X_i\Big\}$$

$$+ e \otimes \sum_T \Big\{\prod_{i \in T} R[X_i] \otimes X/\prod_{i \in T} X_i\Big\} \quad \text{and} \quad (A.3)$$

$$\sum_{T,T'} \Big\{\prod_{i \in T} R[X_i] \otimes \Big\{\prod_{j \in T'} R[\{X/\prod_{i \in T} X_i\}_j] \otimes$$

$$\otimes \{X/\prod_{i \in T} X_i\}/\prod_{j \in T'} \{X/\prod_{i \in T} X_i\}_j\Big\}\Big\}. \quad\quad (A.4)$$

We investigate $\{X/\prod_{i \in T} X_i\}_j$. Either this is a PSW of X or not. If not there must exist a PSW X_m of X and some PSWs X_k with $k \in T^m \subset T$ such that $\{X/\prod_{i \in T} X_i\}_j = X_m/\prod_{k \in T^m} X_k$. This means that $T' = T_1 \oplus T_2$ (both T_1, T_2 can be empty but not the sum) and

$$\prod_{j \in T'} R[\{X/\prod_{i \in T} X_i\}_j] = \prod_{l \in T_1} R[X_l] \prod_{m \in T_2} R[X_m/\prod_{k_m \in T^m} X_{k_m}].$$

Let us assume that T_2 contains at least two elements m_1, m_2 and perform the factorization

$$\{X/\prod_{i \in T} X_i\}/(\{X_{m_1}/\prod_{k_1 \in T^{m_1}} X_{k_1}\}\{X_{m_2}/\prod_{k_2 \in T^{m_2}} X_{k_2}\}). \quad (A.5)$$

Recall that $T^{m_1} \subset T$ and $T^{m_2} \subset T$ and assume that $X_n \in T^{m_1} \cap T^{m_2}$. The fraction (A.5) will only be non-zero if $X_{m_1}/\prod_{k_1 \in T^{m_1}} X_{k_1}$ and $X_{m_2}/\prod_{k_2 \in T^{m_2}} X_{k_2}$ occur together and disjoint in at least one row of $X/\prod_{i \in T} X_i$. These rows correspond to those rows of X each of which contain all X_i, $i \in T$, too. But each X_i occurs precisely once in any row, so does the X_n in question, hence it will either occur in T^{m_1} or in T^{m_2}, but never in both. Therefore, we have a direct sum decomposition $T = T_3 \oplus \bigoplus_{m \in T_2} T^m$

and (A.4) takes the form

$$
\begin{aligned}
\text{(A.4)} \quad = & \sum_{\{T_1,T_2,T_3,\bigcup_{m\in T_2}T^m\}} \Big\{ \prod_{i\in T_3}R[X_i] \prod_{m\in T_2}\{ \prod_{k_m\in T^m}R[X_{k_m}] \} \\
& \otimes \prod_{l\in T_1}R[X_l] \prod_{m\in T_2}R[X_m/\prod_{k_m\in T^m}X_{k_m}] \\
& \otimes X/(\prod_{m\in T_2}X_m \prod_{l\in T_1}X_l \prod_{i\in T_3}X_i) \Big\} \\
= & \sum_T \Big\{ \{ \sum_{T_3\subset T}\prod_{i\in T_3}R[X_i] \otimes \sum_{T_1\subset T/T_3}\prod_{l\in T_1}R[X_l] \} \\
& \times \prod_{m\in T_2=T/(T_1\oplus T_3)}\{\sum_{T^m}\{ \prod_{k_m\in T^m}R[X_{k_m}] \\
& \otimes R[X_m/\prod_{k_m\in T^m}X_{k_m}] \}\} \otimes X/\prod_{j\in T}X_j \Big\}.
\end{aligned}
\tag{A.6}
$$

Note that T_1, T_2, T_3 can be empty, in that case the missing product over $R[X_j]$ has to be replaced by e. If T_2 is empty then the sum over $T_1 = T/T_3$ has to be omitted. Observe that neither $T_1 \oplus T_2$ nor $T_3 \oplus T_2$ can be empty, but these two terms $T_2 = \emptyset$ and either $T_1 = \emptyset$ or $T_3 = \emptyset$ are precisely those of (A.3). All together can be rewritten as

$$
\begin{aligned}
& \text{(A.3)} + \text{(A.4)} \\
= & \sum_T \Big\{ \prod_{j\in T}\{ e\otimes R[X_j] + R[X_j]\otimes e \\
& + \sum_{T^j}\{ \prod_{k_j\in T^j}R[X_{k_j}] \otimes R[X_j/\prod_{k_j\in T^j}X_{k_j}] \}\} \otimes X/\prod_{j\in T}X_j \Big\} \\
= & (\Delta\otimes\text{id})\big[\sum_T \{ \prod_{j\in T}R[X_j] \otimes X/\prod_{j\in T}X_j \}\big],
\end{aligned}
\tag{A.7}
$$

and we conclude

$$
\begin{aligned}
\text{(A.2)} + \text{(A.3)} + \text{(A.4)} & = (\Delta\otimes\text{id})\circ\Delta[X] \\
& = (\text{id}\otimes\Delta)\circ\Delta[X].
\end{aligned}
\tag{A.8}
$$

To finish the proof of coassociativity of Δ we must write down

$$
\begin{aligned}
(\text{id}\otimes\Delta)\circ\Delta[R[X]] & = (\text{id}\otimes\Delta)\circ(\text{id}\otimes R')\circ\Delta[X] \\
& = (\text{id}\otimes\text{id}\otimes R')\circ(\text{id}\otimes\Delta)\circ\Delta[X] \\
& = (\text{id}\otimes\text{id}\otimes R')\circ(\Delta\otimes\text{id})\circ\Delta[X] \\
& = (\Delta\otimes\text{id})\circ\Delta[R[X]],
\end{aligned}
$$

$$
\begin{aligned}
(\text{id}&\otimes\Delta)\circ\Delta[XY] \\
& = \hat{m}\big[\{(\text{id}\otimes\Delta)\circ\Delta[X]\} \otimes \{(\text{id}\otimes\Delta)\circ\Delta[Y]\}\big] \\
& = \hat{m}\big[\{(\Delta\otimes\text{id})\circ\Delta[X]\} \otimes \{(\Delta\otimes\text{id})\circ\Delta[Y]\}\big] \\
& = (\Delta\otimes\text{id})\circ\Delta[XY].
\end{aligned}
$$

We have defined $\hat{m}[\{X'\otimes X''\otimes X'''\} \otimes \{Y'\otimes Y''\otimes Y'''\}] := X'Y' \otimes X''Y'' \otimes X'''Y'''$ as well as $R'[e] = e$ and $R'[X] = R[X]$ for $X \neq e$. $\square$

Proposition 2 *The 'antipode' S fulfills $m\circ(S\otimes\text{id})\circ\Delta \approx E\circ\varepsilon \approx m\circ(\text{id}\otimes S)\circ\Delta$, and on PWs X not containing R we even have $m\circ(S\otimes\text{id})\circ\Delta[X] \sim 0 = E\circ\varepsilon[X]$.*

Proof. The case $X = e$ is trivial. Let $X \neq e$ be an iPW, which is not $R[X']$:

$$
\begin{aligned}
m\circ & (S\otimes\text{id})\circ\Delta[X] \\
& = m\circ(S\otimes\text{id})[e\otimes X + R[X]\otimes e + P_2\Delta[X]] \\
& = X + S[R[X]] + m\circ(S\otimes\text{id})\circ P_2\Delta[X] \\
& = X - R[X + m\circ(S\otimes\text{id})\circ P_2\Delta[X]] \\
& \quad + m\circ(S\otimes\text{id})\circ P_2\Delta[X] \\
& = (\text{id} - R)[X + m\circ(S\otimes\text{id})\circ P_2\Delta[X]] \\
& \sim 0 = E\circ\varepsilon[X],
\end{aligned}
$$

$$
\begin{aligned}
m\circ & (\text{id}\otimes S)\circ\Delta[X] \\
& = m\circ(\text{id}\otimes S)[e\otimes X + R[X]\otimes e + P_2\Delta[X]] \\
& = S[X] + R[X] + m\circ(\text{id}\otimes S)\circ P_2\Delta[X] \\
& = -(X + m\circ(\text{id}\otimes S)\circ P_2\Delta[X]) + R[X] \\
& \quad + m\circ(\text{id}\otimes S)\circ P_2\Delta[X] \\
& = -(\text{id} - R)[X] \sim 0 = E\circ\varepsilon[X].
\end{aligned}
$$

As we have chosen (20b), we must also compute (X is again an iPW)

$$
\begin{aligned}
m\circ & (S\otimes\text{id})\circ\Delta[R[X]] \\
& = m\circ(S\otimes\text{id})[e\otimes R[X] + R[X]\otimes e + P_2\Delta[R[X]]] \\
& = R[X] + S[R[X]] + m\circ(S\otimes\text{id})\circ P_2\Delta[R[X]] \\
& = R[X] - R[X + m\circ(S\otimes\text{id})\circ P_2\Delta[X]] \\
& \quad + m\circ(S\otimes\text{id})\circ P_2\Delta[R[X]] \\
& = (m\circ(\text{id}\otimes R) - R\circ m)[(S\otimes\text{id})\circ P_2\Delta[X]] \\
& \approx 0 = E\circ\varepsilon[R[X]].
\end{aligned}
$$

We need condition (18) in the form $R\circ m = R\circ m\circ(\text{id}\otimes R)$ to have equivalence. The remaining case is more complicated:

$$
\begin{aligned}
m\circ & (\text{id}\otimes S)\circ\Delta[R[X]] \\
& = S[R[X]] + R[X] + m\circ(\text{id}\otimes S)\circ P_2\Delta[R[X]] \\
& = -R[X + m\circ(S\otimes\text{id})\circ P_2\Delta[X]] + R[X] \\
& \quad + m\circ(\text{id}\otimes S)\circ P_2\Delta[R[X]] \\
& = m\circ(\text{id}\otimes S)\circ P_2\Delta[R[X]] \\
& \quad - R[m\circ(S\otimes\text{id})\circ P_2\Delta[X]].
\end{aligned}
\tag{A.9}
$$

We transform the first term, using the definition of S acting on $R[.]$:

$$
\begin{aligned}
m\circ & (\text{id}\otimes S)\circ P_2\Delta[R[X]] \tag{A.10} \\
= & -m[P_2\Delta[R[X]]] \\
& - m\circ(\text{id}\otimes\{R\circ m\circ(S\otimes\text{id})\circ P_2\Delta\})\circ P_2\Delta[X] \\
= & (R\circ m - m\circ(\text{id}\otimes R))\circ(P_2\Delta[X] \\
& + (\text{id}\otimes\{m\circ(S\otimes\text{id})\circ P_2\Delta\})\circ P_2\Delta[X]) \\
& - R\Big[m[P_2\Delta[X]] \\
& + m\circ(\text{id}\otimes m)\circ(\text{id}\otimes S\otimes\text{id})\circ(\text{id}\otimes P_2\Delta)\circ P_2\Delta[X]\Big].
\end{aligned}
$$

Now observe that due to coassociativity of Δ we have

$$
(\text{id}\otimes P_2\Delta)\circ P_2\Delta[X] = P_3\circ(\text{id}\otimes\Delta)\circ\Delta[X]
$$

$$= P_3 \circ (\Delta \otimes \mathrm{id}) \circ \Delta[X]$$
$$= (P_2 \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ P_2 \Delta[X] \; ,$$

with $P_3 = (\mathrm{id}-E\circ\varepsilon) \otimes (\mathrm{id}-E\circ\varepsilon) \otimes (\mathrm{id}-E\circ\varepsilon)$. Note that Δ is multiplicative, not $(P_2\Delta)$. Using also associativity of m we can write

$$- R\big[m\circ(\mathrm{id} \otimes m)\circ(\mathrm{id} \otimes S \otimes \mathrm{id})\circ(\mathrm{id} \otimes P_2\Delta)\circ P_2\Delta[X]\big]$$
$$= -R\big[m \circ (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes S \otimes \mathrm{id}) \circ$$
$$\circ(P_2 \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ P_2\Delta[X]\big] \; .$$

We have computed $(\Delta \otimes \mathrm{id}) \circ P_2\Delta[X]$ in (A.7). By inspection of that formula we find that $(P_2\otimes\mathrm{id})\circ(\Delta\otimes\mathrm{id})\circ P_2\Delta[X]$ equals $(\Delta \otimes \mathrm{id}) \circ P_2\Delta[X] - $(A.3), which gives

$$-R\big[m \circ (\mathrm{id} \otimes m) \circ (\mathrm{id} \otimes S \otimes \mathrm{id}) \circ (\mathrm{id} \otimes P_2\Delta) \circ P_2\Delta[X]\big]$$
$$= -R\Big[\sum_T m\Big[\prod_{j\in T} \{m \circ (\mathrm{id} \otimes S) \circ \Delta[R[X_j]]\} \otimes$$
$$\otimes X/\prod_{j\in T}X_j\Big]\Big]$$
$$+ R\Big[\sum_T m\Big[\prod_{j\in T} S[[R[X_j]]] \otimes X/\prod_{j\in T}X_j\Big]\Big]$$
$$+ R\Big[\sum_T m\Big[\prod_{j\in T}[R[X_j]] \otimes X/\prod_{j\in T}X_j\Big]\Big] \; .$$

The last term cancels $-R[m[P_2\Delta[X]]]$ in (A.10) and the middle term cancels $-R[m \circ (S \otimes \mathrm{id}) \circ P_2\Delta[X]]$ in (A.9). We end up with the same problem as before, to calculate $m\circ(\mathrm{id}\otimes S)\circ\Delta[R[X_i]]$, however, these X_i are *smaller* than the original X. This leads to an iteration which stops if X_i is primitive, and for primitive X_i we have

$$m \circ (\mathrm{id} \otimes S) \circ \Delta[R[X_i]] = S[R[X_i]] + R[X_i] = 0 \; .$$

The conclusion is that it is condition (18) required in (A.10) which separates us from zero: $m\circ(\mathrm{id}\otimes S)\circ\Delta[R[X]] \approx 0 = E \circ \varepsilon[R[X]]$ for all iPW X.

It remains to apply $m \circ (\mathrm{id} \otimes S) \circ \Delta$ and $m \circ (S \otimes \mathrm{id}) \circ \Delta$ to products $X = \prod_i X_i \prod_j R[Y_j]$. Here we have the multiplicativity of Δ (24) and S (25b) at disposal, so we clearly get

$$m \circ (\mathrm{id} \otimes S) \circ \Delta[X] \approx 0 = E \circ \varepsilon[X]$$
$$\approx m \circ (S \otimes \mathrm{id}) \circ \Delta[X] \; . \text{(A.11)}$$

One case however is special. For $X = \prod X_i$, where none of the X_i is $R[X'_i]$, we have

$$m \circ (S \otimes \mathrm{id}) \circ \Delta[\prod_i X_i]$$
$$= \prod_i \big\{(\mathrm{id} - R)\big[X_i + m \circ (S \otimes \mathrm{id}) \circ P_2\Delta[X_i]\big]\big\}$$
$$\sim 0 = E \circ \varepsilon[\prod_i X_i] \; . \tag{A.12}$$

The reason is that $(\mathrm{id} - R)\big[X_i + m \circ (S \otimes \mathrm{id}) \circ P_2\Delta[X_i]\big]$ is *convergent as it reproduces the forest formula*, see (27a). Now, multiplication of $(\mathrm{id}-R)\big[X_i + m\circ(S\otimes\mathrm{id})\circ P_2\Delta[X_i]\big]$ by a convergent term is equivalent to zero. It is even strongly equivalent ($\simeq$) to zero which means that the integral is *finite*. On the other hand, $m\circ(\mathrm{id}\otimes S)\circ\Delta[\prod_i X_i] = \prod_i \big\{(R - \mathrm{id})[X_i]\big\}$ is a product of divergent terms, so we need (18) in this case to obtain equivalence to zero. The fact that $m\circ(S\otimes\mathrm{id})\circ\Delta$ gives the forest formula is essential for (A.12) holding in any renormalization scheme.

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